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SPATIAL MARCHING TECHNIQUES FOR APPROXIMATE SOLUTIONS TO THE HYPERSONIC BLUNT BODY PROBLEM

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SUMMARY

Two reduced forms of the Euler equations which allow spatial marching in subsonic flow regions are investigated for solving the inviscid blunt body problem. An analysis of the eignevalues to determine the properties of the steady and unsteady forms of the governing equations is performed. The steady forms of both the thin shock layer equations and the pressure gradient splitting method are appropriate for a marching solution technique. Numerical results from the thin layer equations are less accurate, and the suitability of this approach in a global iteration procedure is questioned as the analysis shows information is not transmitted upstream. The pressure gradient splitting method gives more accurate results with a single downstream march and appears better suited for use in a global iteration procedure to obtain the complete solution of the Euler equations. Further evaluation of the pressure gradient splitting method shows that it can be readily applied over a range of Mach numbers, and the accuracy of the results is only slightly dependent on the free-stream Mach number.

KEY WORDS Euler equations Spatial marching Subsonic flow Blunt body problem Thin shock layer Pressure gradient splitting

1. INTRODUCTION

This paper is part of an investigation aimed at obtaining solutions to the hypersonic blunt body problem with an efficient computational approach that can be used in design studies. Although the present paper is limited to inviscid flow of a perfect gas, the desired solutions should include viscous and real gas effects. One of the significant phenomena that occurs in modelling hypersonics flows is that air must be treated as a chemically reacting, multi-component gas mixture. Before this complete problem can be handled effectively, the basic numerical solution procedure must be developed and its properties understood.

Temporal marching techniques that solve the unsteady governing equations are one of the standard techniques for solving the blunt body problem. For the inviscid case, the work of Moretti and Bleich[1] is one of the fundamental papers on this approach. There are presently several items that limit the usefulness of this approach: (1) initial conditions must be assumed that are a crude assumption of the flow field or are far from the desired solution, i.e. uniform flow; (2) the convergence of the solution to a steady state is slow with a large number of time steps required, especially for the viscous flow case; (3) for gas models with viscous effects and finite rate chemistry, this approach strains the capabilities of today's supercomputers. It is interesting to observe that

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some of the recent time marching techniques, for example those of Zhuang and Zhang[2] and Candler and MacCormack,[3] are similar to the spatial marching techniques discussed in this paper.

Several spatial marching approaches have been used previously for solving the hypersonic flow over a blunt body. The initial inviscid work of Chernyi,[4] Chester[5] and Freeman[6] used the Newtonian limit equations obtained by expanding about $\gamma = 1$ and infinite free-stream Mach number. Results from this approach are of limited accuracy as the tangential velocity is zero at the surface and a singularity occurs on a sphere at 60°. More accurate results are obtained from the thin shock layer equations (convective terms neglected in normal momentum equation) which were developed for solving viscous flows over bodies at low Reynolds numbers by Cheng.[7] This approach was extended by Davis[8] to a composite set of equations that includes all the terms in the Euler equations plus the additional terms that occur in the second-order boundary layer equations and is referred to as the viscous shock layer method.

Another marching approach has evolved from the initial work of Rudman and Rubin[9] for hypersonic flow over a flat plate where the pressure gradient in the flow direction is neglected in the momentum equation along the plate.In addition, the viscous terms with second derivatives in the flow direction are neglected in the Navier–Stokes equations. This reduced form of the Navier–Stokes equations with some type of approximation for the pressure gradient in the flow direction has become known as the parabolized Navier–Stokes (PNS) approach. To take into account the pressure gradient more accurately, Lin and Rubin[10] introduced the sublayer approximation for calculating supersonic flow over a cone. In this approach the pressure gradient in the subsonic flow region near the body surface is determined from the pressure gradient in the adjacent supersonic region. The sublayer approximation gives a marching solution technique if the sublayer pressure gradient is evaluated sufficiently far into the supersonic region. Lubard and Helliwell[11,12] have shown that a stable numerical marching scheme is obtained with the complete pressure gradient term retained if the marching step size is greater than a critical value. If part of the pressure gradient is retained in the subsonic flow regions, the governing equations can still be solved with spatial marching. The part of the pressure gradient that can be retained in the governing equations to avoid elliptic governing equations in the subsonic flow regions was determined by Vigneron *et al.*[13] This result shows how to split the pressure gradient into two parts; one part is included in the marching solution, while the other part is neglected or evaluated with a global iteration procedure. Additional contributions have been made to the PNS approach by Schiff and Steger,[14] whose technique was developed for governing equations in general co-ordinates and in strong conservation law form. The use of the PNS equations has become the standard technique to predict flow downstream on bodies where the inviscid flow is supersonic or hypersonic. More details on these approaches as they apply to the Euler equations will be given in Section 3.

The reduction of the Navier–Stokes equations to a system of governing equations that can be solved with a marching scheme has been developed for various flow problems. A review of some of this work is available in Davis and Rubin,[15] while techniques for internal flows are given in Blottner.[16] More complete solutions of the Navier–Stokes equations can be obtained with the spatial marching procedure when a global iteration for the pressure field is included. A summary of some of this work is given in Rubin and Reddy.[17]

The present investigation is concerned with numerical solutions of the thin shock layer equations and the Euler equations with pressure gradient splitting for solving the inviscid blunt body problem. These approximate solutions can provide useful engineering predictions or can be used as initial conditions in a global iteration procedure which will provide the numerical solution to the complete governing equations. With an assumed shock wave, these approaches allow the solution to be started at the stagnation point and marched downstream through the subsonic flow region. Beyond the sonic line the complete Euler equations are solved. At the stagnation point the governing equations become ordinary differential equations with an approximation for the second derivative of the pressure gradient in the tangential direction required. The pressure gradient term is evaluated from the normal momentum equation with the thin shock layer approach, while the pressure gradient at the shock wave is used across the layer with the pressure gradient splitting approach. In both cases the governing equations are approximated with a box scheme, and the difference equations are locally linearized and solved as a coupled system of equations. At each marching step the solution is iterated to account for the linearization of the governing equations, and the shock wave properties are held fixed in the subsonic region but updated downstream where the flow is completely supersonic.

A numerical procedure has been developed and a code written for solving the aforementioned problem. The details of the development of this code are given in Davis and Blottner[18] and Blottner.[19] Numerical predictions have been made with both methods and are compared with steady-state results obtained from an unsteady solution code. Further work is required to improve the iteration procedure used to move the shock wave.

2. GOVERNING CONSERVATION EQUATIONS

The inviscid flow equations are written in surface co-ordinates with s the distance along the surface from the stagnation point and n the normal distance away from the surface. The velocity components are u and v and are in the s and n co-ordinate directions respectively. The Euler equations are written in the following form:

s-momentum

$$\rho u \frac{\partial u}{\partial s} + H\rho v \frac{\partial u}{\partial n} + \kappa \rho u v + \frac{\partial p}{\partial s} = 0, \tag{1}$$

n-momentum

$$\rho u \frac{\partial v}{\partial s} + H\rho v \frac{\partial v}{\partial n} - \kappa \rho u^2 + H \frac{\partial p}{\partial n} = 0, \tag{2}$$

continuity

$$\frac{\partial (r\rho u)}{\partial s} + \frac{\partial (rH\rho v)}{\partial n} = 0, \tag{3}$$

energy

$$u \frac{\partial H_T}{\partial s} + Hv \frac{\partial H_T}{\partial n} = 0, \tag{4}$$

where the metric term $H = 1 + \kappa n$ and the radial distance $r = r_b + n\cos\theta_b$. The variable κ is the surface curvature, r_b is the radial distance from the body axis to the surface of the body and θ_b is the angle between the body surface and the body axis. The total enthalpy H_T is written in terms of the pressure p, the density ρ and the total energy e, which gives $H_T = (e+p)/\rho = h + \frac{1}{2}(u^2 + v^2)$, where for a perfect gas the specific enthalpy is $h = [\gamma/(\gamma - 1)]p/\rho$.

3. REDUCED GOVERNING EQUATIONS

The foregoing Euler equations are elliptic when the flow is subsonic and hyperbolic for supersonic flow. There are several technniques to change the properties of these equations so that a marching solution technique can be used when the flow is subsonic. The initial efforts in solving the blunt body problem considered the *Newtonian limit equations*[4–6] which are obtained by letting $\gamma \rightarrow 1$ and $M_\infty \rightarrow \infty$. A small parameter $\varepsilon = (\gamma - 1)/(\gamma + 1)$ is introduced and the variables are expanded as follows: $\rho = \rho_0/\varepsilon + \ldots$, $v = \varepsilon v_0 + \ldots$ and $n = \varepsilon N + \ldots$. The reduced Euler equations from the leading terms in the expansion and with the subscripts neglected become

s-momentum

$$\rho u \frac{\partial u}{\partial s} + H \rho v \frac{\partial u}{\partial n} = 0, \tag{5}$$

n-momentum

$$-\kappa \rho u^2 + H \frac{\partial p}{\partial n} = 0, \tag{6}$$

and the mass and energy equations remain the same. For this case the governing equations are parabolic and the shock wave is next to the body surface. Therefore the bow shock location is known and the flow properties behind the shock wave are known from the Rankine–Hugoniot relations. This gives $\rho \rightarrow \infty$, $v \rightarrow 0$ and the u-velocity and enthalpy are constant along streamlines. These equations have a singularity at 60° on a sphere.

The *thin shock layer*[7] or thin viscous shock layer equations are the same as the Newtonian limit equations except $\partial p/\partial s$ is treated as a principal term in the s-momentum equation. The s-momentum, mass and energy equations are the complete equations (1), (3) and (4) given above. The approximation for the n-momentum equation is equation (6), where the convection terms in the complete equation have been neglected. The thin shock layer equations are of initial value type (parabolic) in the s-direction and can be marched spatially. For subsonic flow, the flow variables from the Rankine–Hugoniot relations with specified shock wave are used as boundary conditions. The flow field predicted from these equations has more appropriate behaviour near the body surface and the u-velocity varies along the surface.

In the initial work of Cheng[7] the inverse blunt body problem of a specified shock wave was used and the body shape was obtained from the solution. This approach is truly a spatial marching procedure. In the work of Davis[8] the direct problem is solved where the body shape is specified and the shock wave location is determined with a global iteration procedure. The neglected convective terms in the normal momentum equation are retained when the global iteration is performed. The earlier work with a perfect gas model was modified to include a finite rate chemistry model by Davis and extended further by Moss.[20] An improved iteration technique was developped by Srivastava *et al.*[21] and the latest work on this problem is given in Gordon and Davis.[22] The extension of the viscous shock layer approach to the three-dimensional case was performed by Murry and Lewis.[23] Lewis and co-workers have performed significant work on the viscous shock layer approach and have inccluded various real gas models into the codes. A few iterations are generally used to include the influence of the neglected convection terms. For problems with strong upstream influences, this approach could have slow convergence properties and this is discussed later in this paper.

Another approach is the *pressure gradient splitting technique*[13] where the pressure gradient term $\partial p/\partial \xi$ is modified in the s-momentum equation with the Vigneron condition, which is expressed as

$$\frac{\partial p^{\mathrm{V}}}{\partial \xi} = p_\xi^{\mathrm{V}} = \omega p_\xi + (1 - \omega) p_\xi^0. \tag{7}$$

The pressure gradient p_ξ^0 is assumed known, while the other part of the pressure gradient p_ξ is obtained as part of the solution procedure. It has been shown by Vigneron *et al.*[13] for the

Cartesian form of the Euler equations and by Prabhu and Tannehill[24] for the equations in generalized co-ordinates that ω can be chosen so that the eigenvalues are real and positive as long as there is no reverse flow. For this situation the equations are hyperbolic and can be solved with a marching technique. For the present set of equations with the Mach number $M_u = u/a$ and speed of sound $a^2 = \gamma p/\rho$, the Vigneron condition becomes

$$\omega \leqslant \gamma M_u^2/[1+(\gamma-1)M_u^2], \quad \text{where} \quad 0 < M_u < 1. \tag{8}$$

When the Mach number M_u is zero, the pressure gradient is completely specified by p_ξ^0. When the Mach number $M_u \geqslant 1$, then $\omega = 1$ and the pressure gradient is completely determined from the solution. This approach has been applied to the governing momentum equations written in Cartesian co-ordinate directions and with Cartesian velocity components. Also the approach is used for the downstream region on hypersonic vehicles where the inviscid flow is supersonic or hypersonic. Recently Davis and Blottner[18,19] have used this marching approach for solving the blunt body problem.

4. TRANSFORMED GOVERNING EQUATIONS

The variables are non-dimensionalized with the free-stream velocity V_∞, free-stream density ρ_∞ and reference length L. The pressure is made non-dimensional with $\rho_\infty V_\infty^2$. New co-ordinates are introduced: $\xi = s/L$ and $\eta = (n/L)/F$, where $F = n_{sh}/L$ is the non-dimensional distance from the body to the shock along a surface normal. The non-dimensional transformed governing equations become

ξ-momentum

$$Fu\frac{\partial u}{\partial \xi} + \frac{F}{\rho}\frac{\partial p}{\partial \xi} + \phi\frac{\partial u}{\partial \eta} - \frac{\eta F_\xi}{\rho}\frac{\partial p}{\partial \eta} + Fuv\kappa = 0, \tag{9}$$

η-momentum

$$Fu\frac{\partial v}{\partial \xi} + \phi\frac{\partial v}{\partial \eta} + \frac{H}{\rho}\frac{\partial p}{\partial \eta} - \kappa F u^2 = 0, \tag{10}$$

continuity

$$\frac{\partial (Fr\rho u)}{\partial \xi} + \frac{\partial (r\rho\phi)}{\partial \eta} = 0, \tag{11}$$

energy

$$Fu\frac{\partial H_T}{\partial \xi} + \phi\frac{\partial H_T}{\partial \eta} = 0, \tag{12}$$

where $\phi = vH - u\eta F_\xi$ and the metric term $H = 1 + \kappa\eta F$. The foregoing governing equations can be used to solve for four dependent variables and the ones used in this investigation are the u and v velocity components, the pressure p and the temperature T. The temperature is non-dimensionalized with V_∞^2/c_p and for a perfect gas is obtained from $T = h = [\gamma/(\gamma-1)]p/\rho$. Therefore, wherever the density ρ appears in the governing equations, it is determined from and replaced with this relation.

The governing equations are completed with boundary conditions. At the wall there is one boundary condition, that the normal velocity v is zero. If the shock shape is specified, all of the flow properties u, v, p and T are known behind the shock wave from the Rankine–Hugoniot relations. Since the four governing equations are first-order, four boundary conditions are appropriate, while for the present problem there are five boundary conditions. The extra boundary condition is required to evaluate the shock layer thickness F which is an additional unknown in the governing equations. When F is determined, the specified shock location can be checked to see if the appropriate shape has been chosen. The determination of the shock wave location requires a global iteration procedure when the flow is subsonic and a local iteration at each marching step when the flow is supersonic.

5. CHARACTERISTICS OF STEADY-STATE GOVERNING EQUATIONS

For evaluating the properties of the governing equations, the non-conservation forms of the unsteady equations are usually used with dependent variables density, velocity components and pressure. For the present case the velocity components are tangent and normal to the body surface rather than the Cartesian components. The governing equations are written in matrix notation with the following equation order: (1) continuity, (2) ξ-momentum, (3) η-momentum and (4) pressure. The pressure equation results from a combination of the conservation equations and for the present co-ordinate system has been developed by Blottner.[25] The transformed equations (9)–(12) after expansion, change of order, addition of time-dependent terms and combination of equations become

$$\frac{\partial \mathbf{Q}}{\partial t}+(\mathbf{A}^{+}+\mathbf{A}^{-})\frac{\partial \mathbf{Q}}{\partial \xi}+\mathbf{B}\frac{\partial \mathbf{Q}}{\partial \eta}+\mathbf{C}=\mathbf{0}, \tag{13}$$

where the dependent variable and coefficient matrices are

$$\mathbf{Q}=\begin{bmatrix}\rho\\ u\\ v\\ p\end{bmatrix},\qquad \mathbf{A}^{+}=\frac{1}{H}\begin{bmatrix}u & \rho & 0 & 0\\ 0 & u & 0 & \omega/\rho\\ 0 & 0 & \alpha u & 0\\ 0 & \gamma p & 0 & \Omega u\end{bmatrix},$$

$$\mathbf{A}^{-}=\frac{1}{H}\begin{bmatrix}0 & 0 & 0 & 0\\ 0 & 0 & 0 & (1-\omega)/\rho\\ 0 & 0 & (1-\alpha)u & 0\\ 0 & 0 & 0 & (1-\Omega)u\end{bmatrix},$$

$$\mathbf{B}=\frac{1}{HF}\begin{bmatrix}\bar{\phi} & -\rho f & \rho H & 0\\ 0 & \bar{\phi} & 0 & -f/\rho\\ 0 & 0 & \alpha\bar{\phi} & H/\rho\\ 0 & -\gamma f p & \gamma H p & \bar{\phi}\end{bmatrix},\qquad \mathbf{C}=\frac{1}{H}\begin{bmatrix}\rho c\\ \kappa u v\\ -\kappa u^{2}\\ \gamma p c\end{bmatrix}.$$

The new parameters defined are $\Omega=\gamma-\omega(\gamma-1)$, $c=u(r_{\xi}/r)+\kappa v+(\phi/F)(r_{\eta}/r)$, $f=\eta F_{\xi}$ and $\bar{\phi}=\phi-H\eta F_{t}$. For the steady-state case the governing equations become

$$\mathbf{A}^{+}\frac{\partial \mathbf{Q}}{\partial \xi}+\mathbf{B}\frac{\partial \mathbf{Q}}{\partial \eta}+\mathbf{C}=-\mathbf{A}^{-}\frac{\partial \mathbf{Q}}{\partial \xi}, \tag{14}$$

where $\bar{\phi}=\phi$. For the spatial marching solution of this equation, the right-hand side is assumed to be zero or specified. For the governing equations with pressure gradient splitting, $\alpha=1$ and ω is determined from equation (8); while for the governing equations for the thin shock layer, $\alpha=0$ and $\omega=1$.

Along characteristic curves with slope $\lambda=\mathrm{d}\eta/\mathrm{d}\xi$ in the ξ–η plane, the dependent variables $\mathbf{Q}$ are differentiated and the resulting relations are written in the matrix form

$$\frac{\partial \mathbf{Q}}{\partial \xi}+\lambda\frac{\partial \mathbf{Q}}{\partial \eta}=\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}\xi}. \tag{15}$$

Equations (14) and (15) are combined to obtain the simultaneous linear equations for the derivatives of the dependent variables:

$$\begin{bmatrix} \mathbf{I} & \lambda\mathbf{I} \\ \mathbf{A}^{+} & \mathbf{B} \end{bmatrix}\begin{bmatrix} \partial\mathbf{Q}/\partial\xi \\ \partial\mathbf{Q}/\partial\eta \end{bmatrix}=\begin{bmatrix} \mathrm{d}\mathbf{Q}/\mathrm{d}\xi \\ -\mathbf{C} \end{bmatrix}.$$

This system of equations does not have a unique solution along the characteristics when the determinant of the coefficient matrix vanishes. Therefore the characteristics are determined from

$$\det\begin{bmatrix} \mathbf{I} & \lambda\mathbf{I} \\ \mathbf{A}^{+} & \mathbf{B} \end{bmatrix}=0.$$

Since $\mathbf{A}$ and $\mathbf{B}$ are square matrices, this determinant can be expanded and then rewritten as $\det[\mathbf{B}-\lambda\mathbf{A}^{+}]=0$. The evaluation of this determinant gives

$$(\phi-u\lambda F)^2(\hat{a}\lambda^2-\hat{b}\lambda+\hat{c})=0,$$

where the coefficients are

$$\hat{a}=\alpha(\Omega M_u^2-\omega),$$
$$\hat{b}=\alpha[(1+\Omega)M_\phi M_u+f(1+\omega)]/F,$$
$$\hat{c}=[\alpha(M_\phi^2-f^2)-H^2]/F^2.$$

The following notation is used in the above coefficients: $M_\phi=M_vH-M_uf$, $M_v=v/a$ and $f=\eta F_\xi$. Two eigenvalues are obtained from $\phi-u\lambda F=0$, which gives

$$\lambda_{1,2}=\left(\frac{vH}{u}-f\right)\Big/F. \tag{16}$$

For these eigenvalues the characteristics are streamlines and indicate the equations are hyperbolic in the ξ-direction. The other two eigenvalues are determined from

$$\lambda_{3,4}=[\hat{b}\pm\sqrt{(\hat{b}^2-4\hat{a}\hat{c})}]/2\hat{a}. \tag{17}$$

For the complete *Euler equations*, $\alpha=1$ and $\omega=1$, and the coefficients become $\hat{a}=M_u^2-1$, $\hat{b}=2(f+M_\phi M_u)/F$ and $\hat{c}=(M_\phi^2-f^2-H^2)/F^2$. The properties for this case are determined by

$$\sqrt{(\hat{b}^2-4\hat{a}\hat{c})}=(2H/F)\sqrt{(M_u^2+M_v^2-1)}.$$

If the flow is supersonic ($M_u^2+M_v^2>1$), then the eigenvalues are real and the Euler equations are hyperbolic as is well known. For subsonic flow the eigenvalues are complex and the Euler equations are elliptic.

For the case of *pressure gradient splitting*, initially consider the properties of the governing equations at the surface of the body. At this location, $\eta=0$, $v=0$, $f=0$ and $M_\phi=0$. The coefficients becomes $\hat{a}=\Omega M_u^2-\omega$, $\hat{b}=0$ and $\hat{c}=-1/F^2$. The square root term in the eigenvalue expression (17) becomes

$$\sqrt{(\hat{b}^2-4\hat{a}\hat{c})}=(2/F)\sqrt{\hat{a}}.$$

Therefore, if $\hat{a}>0$, all the eigenvalues will be real and the governing equations are hyperbolic in the ξ-direction. The condition $\hat{a}>0$ or $\omega\leqslant\Omega M_u^2$ is satisfied if the Vigneron condition (8) is satisfied. The properties of the governing equations anywhere in the flow field are now considered. The coefficient $\hat{c}$ is written as

$$\hat{c}=-(H^2-f^2)[1-(M_u^2+M_v^2)]/F^2-(HM_u+fM_v)^2/F^2.$$

Therefore if the flow is subsonic ($M_u^2+M_v^2<1$), then $\hat{c}<0$, $\hat{a}>0$ and the square root term $\sqrt{(\hat{b}^2-4\hat{a}\hat{c})}$ is real and the governing equations are hyperbolic. With this set of equations there is a smooth transition from the reduced form of the governing equations where the flow is subsonic to the complete Euler equations where the flow is supersonic.

For the *thin shock layer* governing equations, the coefficients with $\omega=1$ become $\hat{a}=\alpha(M_u^2-1)$, $\hat{b}=2\alpha\bar{b}$, where $\bar{b}=(f+M_\phi M_u)/F$, and $\hat{c}$ is the same. If the flow is supersonic ($M_u^2+M_v^2>1$), the eigenvalues (17) become

$$\lambda_{3,4}=\{\bar{b}\pm(H/F)\sqrt{[M_v^2+(M_u^2-1)/\alpha]}\}/(M_u^2-1).$$

These eigenvalues are real when $0\leqslant\alpha\leqslant1$. When $\alpha=1$, the complete Euler equations are obtained, while $\alpha=0$ gives the thin shock layer equations and $\lambda_{3,4}\to\pm\infty$. These two eigenvalues indicate the equations have parabolic properties. If the flow is subsonic ($M_u^2+M_v^2<1$), the eigenvalues (17) become

$$\lambda_{3,4}=\{\bar{b}\pm(H/F)\sqrt{[M_v^2-(1-M_u^2)/\alpha]}\}/(M_u^2-1).$$

These eigenvalues are real when $\alpha\leqslant0$, but negative values of α have no physical significance. If α is negative and goes to zero, the thin shock layer equations are obtained and $\lambda_{3,4}\to\pm\infty$. These two eigenvalues indicate the equations have parabolic properties. There can be a jump in the governing equations as the thin shock layer equations are used in the subsonic region, while the complete Euler equations can be used where the flow is supersonic. A smooth transition between the two sets of equations can be obtained only where the flow is supersonic.

6. CHARACTERISTICS OF UNSTEADY GOVERNING EQUATIONS

For a complete solution of the Euler equations with a spatial marching technique and a global iteration to correct for the neglected terms, an investigation of the properties of the unsteady equations (13) is appropriate. In these equations the ξ-derivatives have been split into two parts, where $0\leqslant\omega\leqslant1$ is used in the pressure splitting approach while $\alpha=0$ for the thin shock layer method. The complete Euler equations are obtained with $\omega=1$ and $\alpha=1$ and the governing equations are unsplit. The technique of flux vector splitting has been used in developing upwind difference schemes that take into account the direction information is travelling. The eigenvalues of the matrix $\mathbf{A}^{\pm}$ determine the direction of travel of information in the ξ–t plane. Flux vector or equation splitting is not a unique technique; the form to be used depends on the desired effect. The present motivation is to obtain a system of governing equations that neglect small terms when $\mathbf{A}^-$ is neglected and allow a spatial marching solution. In addition, the retention of the $\mathbf{A}^-$ terms should be handled with an efficient global iteration procedure. The properties of the two previously considered spatial marching methods are investigated by determining the eigenvalues of the coefficient matrix $\mathbf{A}^{\pm}$.

The eigenvalues of $\mathbf{A}^+$ are

$$\lambda_1 = u/H, \qquad \lambda_2 = \alpha u/H,$$

$$\lambda_{3,4} = \frac{1}{2H}\{(1+\Omega)u \pm \sqrt{[(1+\Omega)^2u^2 - 4(\Omega u^2 - \omega\alpha^2)]}\}. \tag{18}$$

When the flow is supersonic, the eigenvalues have the same sign as the u-velocity and information is travelling only in the flow direction. In this case all of the ξ-derivatives should be backward differenced and a spatial marching can be used for the steady-state equations (time step$\rightarrow\infty$). For subsonic flow, information travels upstream which must be suppressed in a spatial marching method. Consider the case where $0 \leqslant u \leqslant a$ and the eigenvalues become:

1. *Complete Euler equations* ($\alpha = 1$ and $\omega = 1$): $\lambda_1 = u/H$, $\lambda_2 = u/H$, $\lambda_3 = (u+a)/H$ and $\lambda_4 = (u-a)/H$. One eigenvalue is negative which indicates information is travelling upstream and at least one ξ-derivative should be forward differenced. A time-like marching procedure is required in this case.
2. *Pressure gradient splitting* ($\alpha = 1$ and $0 \leqslant \omega \leqslant 1$): $\lambda_1 = u/H$, $\lambda_2 = u/H$ and $\lambda_{3,4}$ is given by equation (18). The square root term in equation (18) becomes $\sqrt{[(\Omega-1)^2u^2 + 4\omega a^2]}$ and will be real when $\omega \geqslant 0$. If $\omega \leqslant \Omega(u/a)^2$, which is the Vigneron condition (8), the eigenvalues $\lambda_{3,4}$ will be positive. In this case all of the ξ-derivatives should be backward differenced and a spatial marching can be used for the steady-state equations when the $\mathbf{A}^-$ terms are neglected or assumed known.
3. *Thin shock layer equations* ($\alpha = 0$ and $\omega = 1$): $\lambda_1 = u/H$, $\lambda_2 = 0$, $\lambda_3 = (u+a)/H$ and $\lambda_4 = (u-a)/H$. Eigenvalue λ_4 is negative, which indicates information is travelling upstream and at least one ξ-derivative should be forward differenced. This indicates that the steady-state equations cannot be spatially marched, which is in disagreement with the previous steady-state analysis. Further investigation of the properties of the thin shock layer equations is needed.

Continuing, the eigenvalues of $\mathbf{A}^-$ are

$$\lambda_1 = 0, \qquad \lambda_2 = 0, \qquad \lambda_3 = (1-\alpha)u/H, \qquad \lambda_4 = (1-\Omega)u/H.$$

The eigenvalues λ_3 and λ_4 become, for $0 \leqslant u \leqslant a$:

1. *Complete Euler equations* ($\alpha = 1$ and $\omega = 1$): $\lambda_3 = 0$ and $\lambda_4 = 0$. This term does not contribute for this case.
2. *Pressure gradient splitting* ($\alpha = 1$ and $0 \leqslant \omega \leqslant 1$): $\lambda_3 = 0$ and $\lambda_4 = (1-\Omega)u/H$. One eigenvalue is negative and a forward difference should be used for p_ξ in the ξ-momentum equation. With downstream-to-upstream sweeping in the second step of a global iteration (time marching) procedure, information from downstream is fed throughout the flow field in each iteration.
3. *Thin shock layer equations* ($\alpha = 0$ and $\omega = 1$): $\lambda_3 = 0$ and $\lambda_4 = u/H$. One eigenvalue is positive and a backward difference should be used for v_ξ in the normal momentum equation. The second step of the global iteration does not allow downstream information to feed upstream.

In both of these splittings of the governing equations, the eigenvalues are not split ($\lambda \neq \lambda^+ + \lambda^-$).

From the foregoing eigenvalue analysis of the governing equations, the following comments can be made about the two approximate methods with respect to where to evaluate the derivatives in the flow direction and how a global iteration can be performed:

1. With the pressure gradient splitting approach all of the ξ-derivatives should be backward differenced except for $\mathbf{A}^-(\partial\mathbf{Q}/\partial\xi)$, which is the pressure gradient and is forward differenced. With a global iteration procedure this term allows information to feed upstream. In the

downstream spatial marching solution the term $\mathbf{A}^{-}(\partial\mathbf{Q}/\partial\xi)$ is evaluated from a previous solution or initially is evaluated at the shock wave. Downstream–upstream global iteration procedures for the complete time-dependent equations can be developed with rapid convergence to the solution of the steady Euler equations.

2. With the steady thin shock layer equations all of the ξ-derivatives should be backward differenced. This results in the suppression of all information going upstream. The term $\mathbf{A}^{-}(\partial\mathbf{Q}/\partial\xi)$ or $(u/H)(\partial v/\partial\xi)$ is neglected in the downstream spatial marching solution and should be evaluated with a backward difference when a global iteration is performed. The term $\partial v/\partial\eta$ is also neglected in the normal momentum equation. With a global iteration these terms are evaluated from a previous solution and downstream marching solutions are performed repeatedly. It has not been established that this iteration procedure is stable. Since elliptic partial differential equations are being solved with a unidirectional iteration procedure without information being transmitted upstream, departure solutions are expected with grid refinement and a sufficient number of iterations.

7. NUMERICAL SOLUTION PROCEDURE

The original solution procedure for the Euler equations with pressure gradient splitting was described in Davis and Blottner.[18,19] When further solutions were obtained with fine grids, the results had oscillatory behaviour in the flow direction, with the wavelength long compared with the step size in this direction. The original box finite difference scheme was used for these calculations, where the governing equations are evaluated at the centre of the box and the dependent variables are determined at the corners of the box. The spatial derivatives are evaluated along the edges of the box with two-point difference relations and are averaged to obtain the value at the centre of the box. This is a second-order method in both ξ and η co-ordinate directions. A stability analysis of this scheme with the Vigneron parameter ω determined from equation (8) indicates the scheme is neutrally stable as all of the eigenvalues are one. One approach to remove these oscillations is to multiply the Vigneron parameter ω by a safety factor, SFAC, which is less than one. It appears that as the grid is refined, the value of SFAC decreases and perhaps goes to zero as the marching step size goes to zero. This approach is not a pleasing resolution of the problem.

The approach that is being used to remove the oscillatory behaviour is a first-order box scheme where the governing equations are evaluated at the centre of the upstream edge of the box. Backward differences are used for the ξ-derivatives and these are averaged at the top and bottom edges of the box, while the η-derivatives are evaluated with two-point differences at the downstream edge of the box. This scheme removes the oscillations that occur with the second-order box scheme but is only first-order accurate in the ξ-direction. This first-order box scheme gives poor results near the stagnation point owing to the properties of the continuity equation at this location. Near the stagnation point the flux terms have the following variation away from the origin:

$$Fr\rho u = \alpha\xi^2 + \dots \quad \text{and} \quad \alpha = (FH\rho u_{\xi})_0,$$
$$r\rho\phi = \beta\xi + \dots \quad \text{and} \quad \beta = (H^2\rho v)_0.$$

The continuity equation (11), which is in strong conservation form, requires a second-order scheme to obtain reasonable accuracy and gives at the first step away from the stagnation point the relation $2\alpha + \partial\beta/\partial\eta = 0$. This numerical behaviour is identical to the analytical evaluation. A first-order scheme gives $\alpha+\partial\beta/\partial\eta=0$ and has insufficient accuracy. Therefore the continuity equation is expanded and is written as

$$\frac{\partial(F\rho u)}{\partial\xi}+\frac{\partial(\rho\phi)}{\partial\eta}+\frac{FH\rho u_r}{r}=0, \tag{19}$$

where the relations $r_\eta=F\cos\theta_b$ and $r_\xi=H\sin\theta_b+F_\xi\eta\cos\theta_b$ have been used and where the radial velocity $u_r=u\sin\theta_b+v\cos\theta_b$. Near the stagnation point the variables are expanded as $\sin\theta_b=1-(\xi/R_N)^2/2+\ldots$, $\cos\theta_b=\xi/R_N+\ldots$, $r=H\xi+\ldots$ and $u_r=(u_\xi+v/R_N)\xi+\ldots$. The terms in the continuity equation are expanded as

$$F\rho u=(F\rho u_\xi)\xi+\ldots,$$
$$\rho\phi=\rho vH+\ldots,$$
$$FH\rho u_r/r=F\rho(u_\xi+v/R_N)+\ldots.$$

With the linear or constant behaviour of these terms near the stagnation point, the first-order difference scheme performs much better. Similar behaviour occurs with the term $Fu(\partial v/\partial\xi)$ in the normal momentum equation, but this is a second-order term near the stagnation point. The influence on the results is small and modification of this equation is not necessary.

The limiting form of the continuity equation required at the stagnation point to obtain the initial conditions is

$$\frac{\partial(\rho vH)}{\partial\eta}+2\rho F(u_\xi+v)=0. \tag{20}$$

Equations (19) and (20) are linearized and differenced with midpoint differences as in the previous paper.[19] For the thin shock layer equations, the pressure gradient term in the ξ-momentum equation is treated as an unknown but the convection terms in the normal momentum equation are neglected. The difference equations for both the thin shock layer and the pressure gradient splitting methods are the same, except some terms are neglected and other terms are included in the two approximate solution procedures. The difference in the thin shock layer approach is the governing equation used at the stagnation point to determine $p_{\xi\xi}$. In the pressure gradient splitting method the term $p_{\xi\xi}$ is held constant across the shock layer and equal to the value behind the shock wave. In the thin shock layer approach the term $p_{\xi\xi}$ is evaluated from the normal momentum equation (10). With the expansions $u=u_\xi\xi+\ldots$, $p=p_1+\frac{1}{2}p_{\xi\xi}\xi^2+\ldots$ and $v=v_1+\frac{1}{2}v_{\xi\xi}\xi^2+\ldots$, the normal momentum equation gives the equations

$$\frac{\partial p}{\partial\eta}=-\alpha\rho v\frac{\partial v}{\partial\eta},$$

$$\frac{\partial p_{\xi\xi}}{\partial\eta}=2F\left(\frac{\rho u_\xi}{H}(\kappa u_\xi+\alpha v_{\xi\xi})\right).$$

For the thin layer equations ($\alpha=0$) the pressure is constant across the shock layer at the stagnation point, and $p_{\xi\xi}$ is determined from the second equation and becomes

$$p_{\xi\xi}=(p_{\xi\xi})_{sh}-2F\int_\eta^1(\rho u_\xi/H)(\kappa u_\xi+\alpha v_{\xi\xi})\mathrm{d}\eta. \tag{21}$$

With global iteration the term neglected in the above equation can be included to obtain an improved solution.

8. NUMERICAL RESULTS

The two approximate solution techniques have been used to obtain the inviscid flow over a sphere with free-stream Mach number $M_\infty = 8$. The polynomial expression

$$F = F_0 + \tfrac{1}{2}F_2\xi^2 + \tfrac{1}{4}F_4\xi^4 + \tfrac{1}{6}F_6\xi^6 + \dots$$

has been used to specify the shock layer thickness F, and details of this approach are described in References 18 and 19. The value of F_0 is obtained as part of the solution. The three remaining coefficients are specified and have been adjusted such that the assumed shock location or slope $dF/d\xi$ closely matches the value obtained from the approximate solution. The shock layer thicknesses obtained from the thin shock layer and the pressure gradient splitting approaches are compared with the solution for the complete Euler equations in Figure 1. This solution is obtained from a time marching code[26] and is an accurate numerical solution for the complete Euler equations. The two approximate methods give a thicker shock layer near the stagnation point and approach the exact solution downstream. The pressure gradient splitting method prediction of F approaches the exact solution more rapidly than the thin shock layer result. This is expected as the pressure gradient splitting method smoothly includes more of the neglected pressure gradient terms as the solution is marched downstream. In both cases the complete equations are solved when the flow has become supersonic.

The body surface pressure ratio p_b/p_∞ along the sphere is presented in Figure 2. Again the two approximate solutions are compared with the Euler solution. The pressure gradient splitting method gives values very close to the correct variation, while the thin shock layer approach gives a poorer prediction.

Since the pressure gradient splitting method produces better approximate solutions, and since a global iteration procedure is feasible with this method as the previous eigenvalue analysis indicated, a further evaluation of this method is performed. It is clear from the results presented in Figure 1 that the shock layer thickness near the stagnation point obtained from the reduced Euler

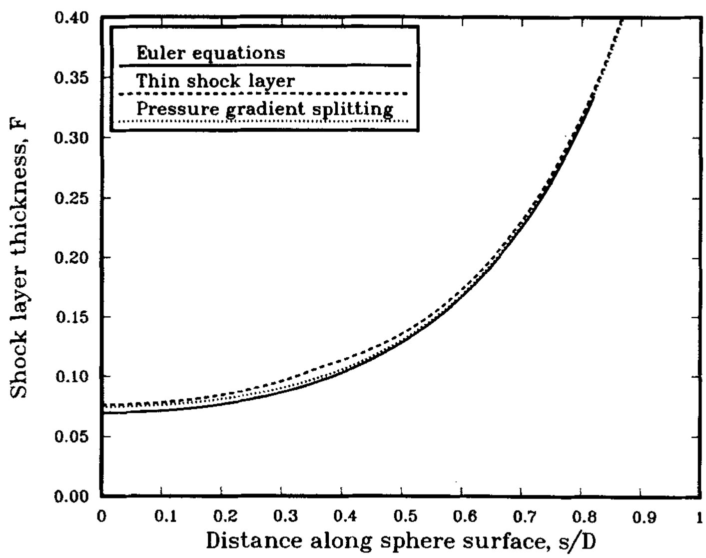

Figure 1. Shock layer thickness F obtained from the numerical solution of the reduced governing equations and from an unsteady Euler code; hypersonic inviscid flow over a sphere with $M_\infty = 8$

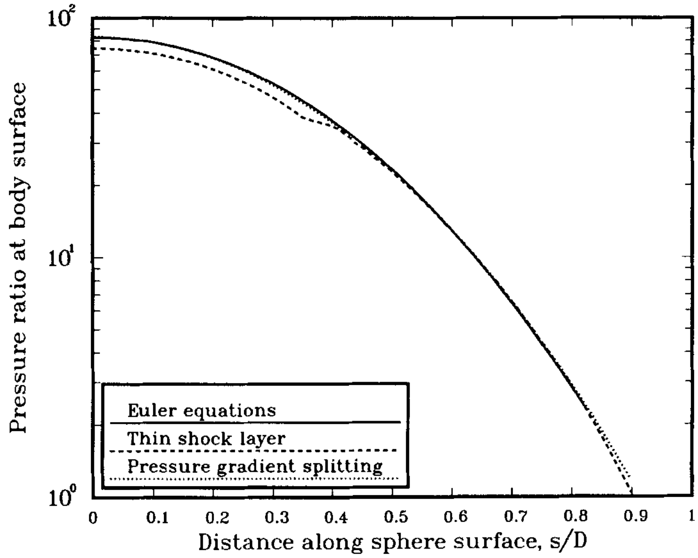

Figure 2. Body surface pressure ratio p_b/p_∞ obtained from the numerical solution of the reduced governing equations and from an unsteady Euler code; hypersonic inviscid flow over a sphere with $M_\infty = 8$

equations has a noticeable error. The shock wave slope relative to the body surface, $dF/d\xi$, predicted by the pressure gradient splitting method also has a noticeable error as shown in Figure 3. The errors for both F and $dF/d\xi$ vary along the body surface, with the maximum error for either being approximately 7%. The flow properties behind the shock wave are determined from the Rankine–Hugoniot relations and these properties are a function of F and $dF/d\xi$. The flow properties behind the shock wave are weakly dependent upon F and are predicted reasonably

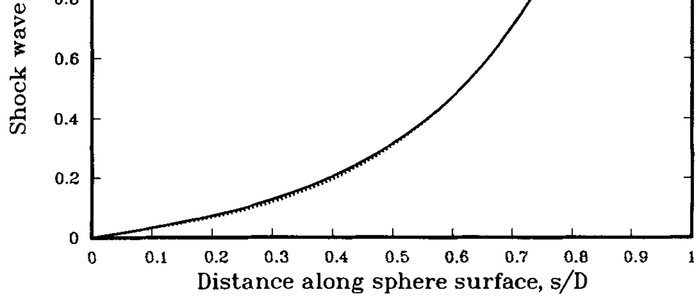

Figure 3. Shock wave slope relative to the body surface, $dF/d\xi$, from the pressure gradient splitting method and from an unsteady Euler code; hypersonic inviscid flow over a sphere with $M_\infty = 8$

well with the pressure gradient splitting method. This behaviour is illustrated in Figure 4, where the pressure behind the shock wave, p_{sh}/p_{∞}, from the pressure gradient splitting method is compared with the value obtained from the complete Euler equation solution. The maximum error of the pressure behind the shock wave is approximately 1·5%. The pressure gradient splitting method provides a reasonably accurate prediction of the pressure field with one downstream marching solution. This pressure result can provide a reasonably accurate estimate of the pressure gradient terms that have been neglected. Therefore a significantly improved prediction of the shock layer thickness should be obtained with only one additional global iteration.

Solutions have been obtained for a range of Mach numbers to determine the properties of the pressure gradient splitting method with a change in flow conditions. The shock wave has been determined for several values of the free-stream Mach number. The values of the coefficients are given in Table I. By interpolation of the tabulated coefficients, the shock wave shape at other free-stream Mach numbers can be estimated. The solutions obtained with the pressure gradient splitting method for the various Mach numbers are illustrated in Figure 5. The reduced Euler equation results for the body surface pressure are compared with the complete Euler equation solutions[27] in this figure. The reduced Euler equation results are close to the Euler results for the range of Mach numbers investigated. The influence of Mach number on the shock wave location at the stagnation point is shown in Figure 6. The pressure gradient splitting method overpredicts

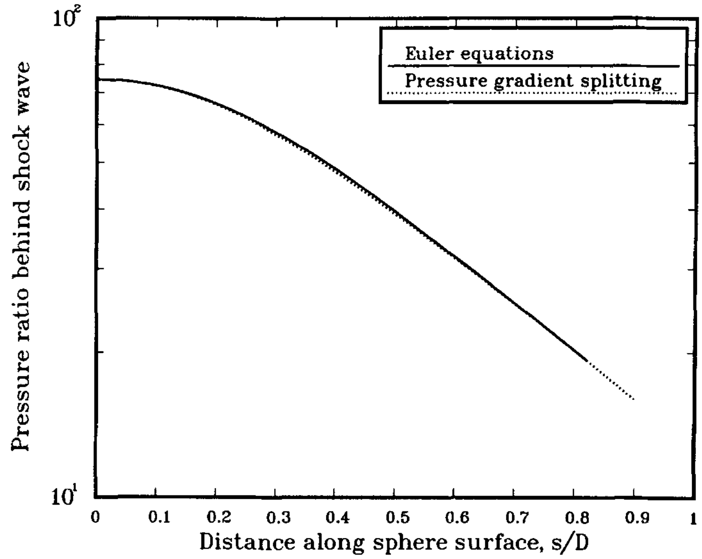

Figure 4. Pressure ratio behind the shock wave, p_{sh}/p_{∞}, from the pressure gradient splitting method and from an unsteady Euler code; hypersonic inviscid flow over a sphere with $M_{\infty}=8$

Table I

M_{∞}	F_0	F_2	F_4	F_6
3	0·1175	0·530	1·25	1·8
5	0·08465	0·355	0·99	1·8
8	0·07465	0·295	0·90	1·8
20	0·06923	0·265	0·83	1·9

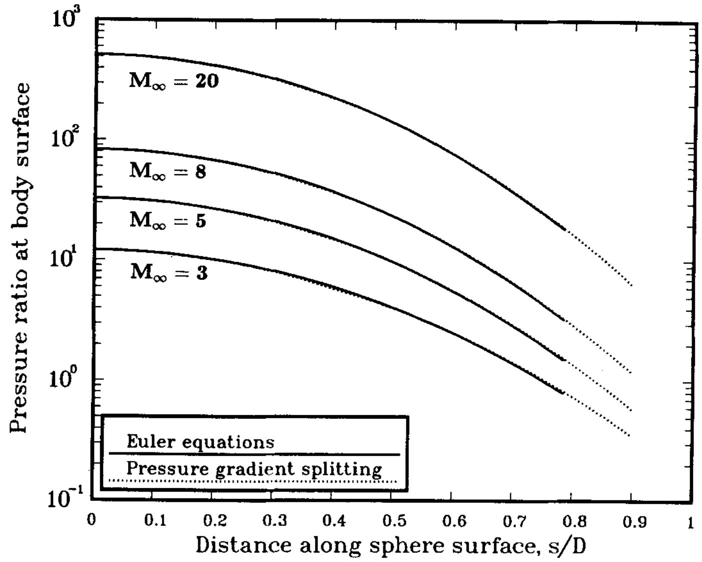

Figure 5. Body surface pressure ratio p_b/p_∞ from the pressure gradient splitting method and from an unsteady Euler code; inviscid flow over a sphere with various M_∞

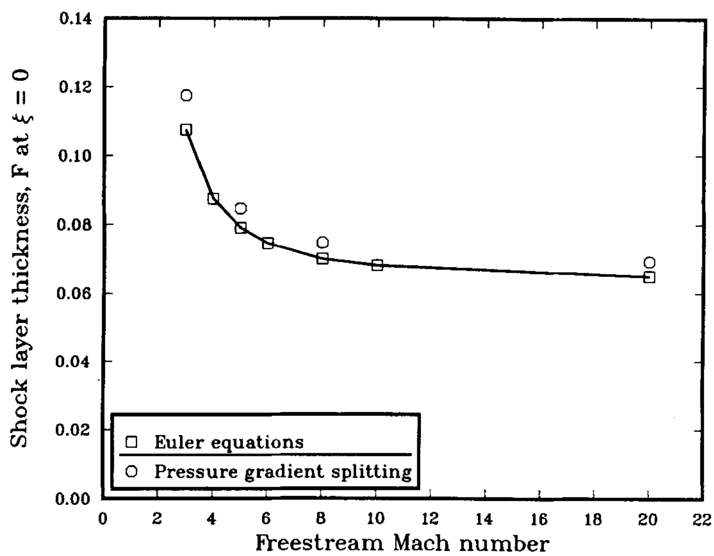

Figure 6. Shock layer thickness at the stagnation point obtained from the pressure gradient splitting method and from an unsteady Euler code; inviscid flow over a sphere with various M_∞

the shock layer thickness for all Mach numbers, with slightly increased error in the shock wave location occurring at lower Mach numbers. With the pressure gradient splitting method, solutions at various Mach numbers are obtained readily and the accuracy of the predictions is only slightly influenced by changing the free-stream Mach number.

9. CONCLUSIONS

The pressure gradient splitting method provides reasonable engineering predictions of the hypersonic flow over blunt bodies. Global iteration to include the complete pressure gradient is feasible and logical from an analysis of the eigenvalues of the governing equations. The thin shock layer equation results are not in as good agreement with the exact solution as the pressure gradient splitting results. In addition, the global iteration of the thin shock layer equations does not provide a mechanism for information to be fed upstream as the eigenvalue analysis shows. This property of the thin layer equations raises questions on the suitability of these equations with an iteration technique for obtaining the complete solution of the Euler equations.

Although the shock layer thickness and shock wave slope predicted with the pressure gradient splitting method are in error near the stagnation point, the flow properties behind the shock wave have a smaller error. The pressure gradient splitting method provides a good initial estimate of the pressure both at the shock wave and the body surface. In addition, it has been shown that the pressure gradient splitting method can be applied over a range of free-stream Mach numbers with the accuracy of the results only slightly dependent on the Mach number.

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